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Wave Chaos in Quantum Systems with Point Interaction

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We study perturbations $\hat{H}$ of the quantized version $\hat{H}_0$ of integrable Hamiltonian systems by point interactions. We relate the eigenvalues of $\hat{H}$ to the zeros of a certain meromorphic function ξ. Assuming the eigenvalues of $\hat{H}_0$ are Poisson distributed, we get detailed information on the joint distribution of the zeros of ξ and give bounds on the probability density for the spacings of eigenvalues of $\hat{H}$. Our results confirm the "wave chaos" phenomenon, as different from the "quantum chaos" phenomenon predicted by random matrix theory.

KEY WORDS: Wave chaos; quantum chaos; point interactions; quantization; spacing of eigenvalues; random matrix theory; integrable systems; Poisson level statistics; Wigner statistics; solvable models of quantum mechanics.

1. INTRODUCTION

Quantum systems exhibiting chaotic behavior ("wave chaos," "quantum chaotic systems") have been intensively studied during the last 10–15 years. The basic question concerning the relation in behavior of classical ergodic systems and quantum systems was already raised by A. Einstein in 1917. Also, in the newer investigations one basic question has been to what extent the quantum mechanical systems reflect the chaotic behavior of the corresponding classical counterparts. Numerical and experimental evidence indicates that a typical chaotic quantum system has local spectral statistical properties (distribution of eigenvalues) which are similar to those of certain

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statistical ensembles of matrices, described by "random matrix theory" (RMT) (see, e.g., refs 8 and 11 for the latter theory, and, e.g., refs. 5 and 6 for the evidence). More particularly, one studies the behavior of the probabilities $\pi_x^c(k) \equiv l(A_k^c(x))/x$, as $0 \leqslant x \to \infty$, $A_k^c(x) \equiv \{E \leqslant x \mid (E, E+c)$ contains k eigenvalues of the quantum (positive) Hamiltonian$\}$, $k \in \mathbb{N} \cup \{0\}$, $c > 0$, l being Lebesgue measure.[13] The behavior of $\pi_x^c(k)$ as $x \to \infty$ found for these "chaotic systems" is in sharp contrast wih the one, based again on numerical and experimental evidence, of the local spectral statistics $\pi_x^c(k)$ of systems which are the quantized version of classical integrable systems; in fact, in the latter systems the above evidence is for a limit Poisson distribution $\pi^c(k) = e^{c\rho}(c\rho)^k/k!$ (for a certain positive constant ρ). Such a Poisson distribution expressing "absence of level repulsion" is very different from the distributions ("Wigner distributions") found in random matrix theory (which yield "level repulsion") and this has been taken as a basic tool for recognizing whether a given quantum system has a chaotic or rather integrable classical underlying counterpart; see, e.g., refs. 7 and 9.

Unfortunately, the distinction is not yet sustained by suitable mathematical results. Let us mention, however, work which provides some discussion of the problems. Berry[3,4] has heuristically shown that in the limit of very small level spacing, quantum mechanical systems coming from nonintegrable classical systems have energy eigenvalues distributed according to the prediction of RMT.

As for integrable systems, the distribution of eigenvalues is known in several particular cases, and shown to be of the Poisson type (see, e.g., ref. 7); however, it is still an open question whether this holds generically (see also the discussion in ref. 13).

In the last 10 years it has become on the other hand more and more clear, on the level of experimental and numerical evidence, that the correspondence between quantum chaotic systems and the RMT cannot be too precise. In fact, the existence of classical periodic orbits strongly influences the quantum systems, an influence which cannot be detected by the RMT approach.

Usually the quantum chaotic systems which are investigated are obtained by quantizing classical chaotic systems. We introduce, following Seba,[12] a new type of system exhibiting "wave chaos," and constructed mathematically on the basis of a physical intuition. These systems will be shown to have a typical wave-chaotic behavior, and are on the other hand simple enough to be handled mathematically. We will calculate the corresponding joint level distribution and show that it differs from the one obtained in the RMT approach. Using it, we will nevertheless show the existence of the "linear level spacing repulsion" for small spacing.

The paper is organized as follows. In Section 2 we define a family of quantum Hamiltonians $\hat{H}_\Theta$ as perturbations by a point interaction of strength Θ of the quantum Hamiltonian $\hat{H}_0$ corresponding to a two-dimensional completely integrable classical system. We give the resolvent of $\hat{H}_\Theta$ and show that the eigenvalues of $\hat{H}_\Theta$ coincide with the zeros of a certain function $\Lambda(z, \Theta)$ or, equivalently, for $\Theta \neq 0$, the zeros of a certain meromorphic function $z \to \xi(z, \Theta)$. The eigenvalues E_n of $\hat{H}_0$ are poles of $\xi(z, \Theta)$. In Section 3 we assume that the eigenvalues E_n of $\hat{H}_0$ are Poisson distributed. We study a finite sum approximation $\xi_N(z)$ of the function $\xi(z) \equiv -\xi(z, \pi)$, converging pointwise as $N \to \infty$ to $\xi(z)$, and give an explicit formula for the joint distribution of the zeros (roots) of ξ_N (hence obtaining information about the eigenvalues of $\hat{H}_\Theta$). We observe that this distribution differs from the one expected by RMT. In Section 4 we show that, assuming again that the eigenvalues E_n of $\hat{H}_0$ are Poisson distributed, the probability $P(s)$ that some of the spacings between two successive zeros of the above function $\xi(z)$ giving the eigenvalues of the perturbed Hamiltonian $\hat{H}_\pi$ belong to $[s, s+ds]$ satisfies $P(s) \geqslant se^{-s}$ and $s \leqslant P(s) \leqslant (9/4)s$ as $s \downarrow 0$. The latter estimate is in agreement with the prediction of RMT, whereas for $s \to \infty$ one gets disagreement with the latter prediction.

2. CONSTRUCTION OF THE QUANTUM HAMILTONIAN

We start with an *integrable* classical system with Hamiltonian $H_0 = H_0(p; q)$; p and q are the classical momenta and coordinates. Because of integrability, the phase-space trajectory of this system is confined on invaiant tori and we can introduce the action–angle variables $(\mathbf{I}; \boldsymbol{\omega})$.[2,10] Using the coordinates $(\mathbf{I}; \boldsymbol{\omega})$, we can rewrite the Hamiltonian as

$$H_0 = H_0(\mathbf{I}) \tag{2.1}$$

(it does not depend on the angular coordinates $\boldsymbol{\omega}$). One can *quantize* the system by the "EBK method" to obtain the corresponding quantum Hamiltonian as

$$\hat{H}_0 = H_0(\hat{\mathbf{I}}) \qquad \text{in} \quad L^2(T_1 \times \cdots \times T_k) \tag{2.2}$$

(the periodic square-integrable function over $T_1 \times \cdots \times T_k$, with Haar–Lebesgue measure, the T_j, $j = 1,..., k$, being tori identified with $[0, 2\pi]$). Here $\hat{\mathbf{I}}$ are the canonical self-adjoint operators

$$\hat{I}_j = i \frac{\partial}{\partial \omega_j} \tag{2.3}$$

defined with periodic boundary conditions on the domain

$$D(\hat{I}_j) = \{f \in L^2(T_j); f \in AC(T_j) \text{ and } f(0) = f(2\pi)\} \tag{2.4}$$

$AC(T_j)$ are the absolutely continuous complex-valued functions on T_j. $\hat{I}_j$ has discrete spectrum with eigenvalues n_j, $n_j \in \mathbb{Z}$. The eigenvalues of $\hat{H}_0$ are given by real numbers

$$E_{n_1;\dots;n_k} = H_0(n_1, n_2, \dots, n_k); \qquad n_1, \dots, n_k \in \mathbb{Z} \tag{2.5}$$

with $H_0(n_1, n_2, \dots, n_k)$ computable by substituing in the expression for $H_0(\mathbf{I})$ the action variable I_j by the value n_j. Let us now order the numbers $E_{n_1;\dots;n_k}$ with respect to their magnitudes. We will denote these ordered numbers as E_i,

$$E_{i+1} \geqslant E_i$$

As explained in Section 1, it is believed that these ordered numbers E_i "form (for a generic integrable system) a Poisson process" in the sense that the quantity described in Section 1 is a Poisson distribution (see, for instance, ref. 13).

In classical mechanics by perturbing integrable systems by a perturbation which is strong enough, all the invariant tori T_j are destroyed and the phase-space trajectory becomes "chaotic" (the intermediary situation of small perturbation when some of the tori are destroyed and some still exist is described by the KAM theorem; see, e.g., refs. 2 and 10). As mentioned in the introduction, in quantum mechanics one expects a transition from the Poisson level statistics to the Wigner statistics when increasing the strength of the perturbation. Our aim here is to show that this transition occurs indeed when perturbing the integrable Hamiltonian H_0 by a "point interaction." This kind of perturbation has been studied intensively in recent years; see, e.g., ref. 1.

We shall suppose here the system to be two-dimensional in order to simplify notations, but all results hold also for higher dimensions. The Hilbert space is $L^2(T)$, with $T = T_1 \times T_2$ a two-torus. The perturbed operator H_Θ is obtained formally from $\hat{H}_0$ by adding to $\hat{H}_0$ a delta function of (renormalized) strength Θ (point interaction of strength Θ, in the sense of ref. 1). It will be described as a certain self-adjoint extension of the symmetric operator $H_{0,0}$, obtained by restricting $\hat{H}_0$ to the domain

$$D_0 = \{f \in D(\hat{H}_0); f(0, 0) = 0\} \tag{2.6}$$

The deficiency indices of $H_{0,0}$ can be investigated using the method of Krein, as adapted by Zorbas[14] (see, e.g., ref. 1 for a more general description of these techniques). Let us now make the following assumptions about the Green's function of the operator $\hat{H}_0$: There exists a measurable function $G(\omega, \omega', z)$ on $T \times T \times \mathbb{C}$ which satisfies

$$((\hat{H}_0 - z)^{-1} f)(\omega) = \int_T G(\omega, \omega', z) \cdot f(\omega')\, d\omega' \tag{2.7}$$

for all $f \in L^2(T)$, $\mathrm{Im}(z) \neq 0$ (this is, e.g., satisfied if $\hat{H}_0$ is the negative of the Laplacian Δ on T). Following ref. 14, we have the following result.

Lemma 1. (i) Assume that $G(\cdot, 0, \pm i) \notin L^2(T)$. Then the deficiency indices of $H_{0,0}$ are equal to $(0, 0)$.

(ii) Assume $G(\cdot, 0, \pm i) \in L^2(T)$. Then $H_{0,0}$ has deficiency indices $(1, 1)$ and the deficiency subspaces $K^{\pm}$ are

$$K^{\pm} = \{f; f(\omega) = c \cdot G(\omega, 0, \mp i); c \in \mathbb{C}\} \quad \blacksquare$$

It is therefore important to check whether the Green's function $G(\omega, 0, \pm i)$ is quadratic integrable. This can, however, be done by writing the Green's function in action–angle representation,

$$(H_0(\hat{I}_1; \hat{I}_2) - z)^{-1} \tag{2.8}$$

which leads to the following criterion:

$$G(\cdot, 0, \pm i) \in L^2(T) \Leftrightarrow \sum_{n, m \in \mathbb{Z}} \frac{1}{H_0(n, m)^2 + 1} < \infty \tag{2.9}$$

where $H_0(I_1; I_2)$ is the classical Hamiltonian given by (2.1).

Let us now suppose that the Green's function $G(\omega, 0, \pm i)$ is quadratic integrable, i.e., (2.9) holds (again this is, e.g., satisfied if $\hat{H}_0 = -\Delta$). In this case the deficiency indices of $H_{0,0}$ are equal to $(1, 1)$ and we can construct the one-parameter family $\hat{H}_\Theta$, $\Theta \in [0, 2\pi)$, of its self-adjoint extensions.

Theorem 2. The self-adjoint extensions of $H_{0,0}$ are defined by the operators

$$\hat{H}_\Theta = H_0(\hat{I}_1, \hat{I}_2)$$

on

$$D(\hat{H}_\Theta) = \{\Psi = \Phi + c \cdot G(\omega, 0, i) - c \cdot e^{i\Theta} \cdot G(\omega, 0, -i); \Phi \in D_0; c \in \mathbb{C}; \Theta \in [0, 2\pi)\}$$

The family $\hat{H}_\Theta$ represents the perturbed Hamiltonians; Θ is the corresponding coupling constant (strength of the point interaction). $\Theta = 0$ corresponds to free Hamiltonian $\hat{H}_{\Theta=0} = \hat{H}_0$; $\Theta = \pi$ corresponds to infinite coupling.

For the proof see ref. 14.

The eigenvalues of $\hat{H}_\Theta$ can be most easily investigated through the Krein formula, which leads to the following result.

Theorem 3. Let $(\hat{H}_\Theta; \Theta \in [0, 2\pi))$ be the self-adjoint extensions of $H_{0,0}$. Their resolvents are given by the kernels

$$(\hat{H}_\Theta - z)^{-1}(\omega, \omega') = (H_0 - z)^{-1}(\omega, \omega') + \lambda(z, \Theta)\,|G(\omega, 0, z)\rangle\langle G(\omega', 0, z)|, \qquad \operatorname{Im} z \neq 0$$

with

$$\lambda(z, \Theta) = (1 - e^{i\Theta})\left[(i - z)\int_T d\omega'\, G(\omega', 0, z)\cdot G(\omega', 0, i) + e^{i\Theta}(i + z)\int_T d\omega'\, G(\omega', 0, z)\cdot G(\omega', 0, -i)\right]^{-1}$$

and

$$|G(\omega, 0, z)\rangle\langle G(\omega', 0, z)|\, f\rangle(\omega) \equiv G(\omega, 0, z)\int G(\omega', 0, z)\, f(\omega')\, d\omega'$$

for all $f \in L^2(T)$.

It is clear that the eigenvalues of $\hat{H}_\Theta$ coincide with the poles of the function $\lambda(z, \Theta)$. The poles of $\lambda(z, \Theta)$ coincide with the zeros of the function $\Lambda(z, \Theta)$:

$$\lambda(z, \Theta) = (1 - e^{i\Theta})\frac{1}{\Lambda(z, \Theta)} \tag{2.10}$$

with

$$\Lambda(z, \Theta) = (i - z)\int_T d\omega\, G(\omega, 0, z)\, G(\omega, 0, i) + e^{i\Theta}(i + z) \times \int_T d\omega\, G(\omega, 0, z)\, G(\omega, 0, -i) \tag{2.11}$$

Using the resolvent equation, for $\operatorname{Im} z_1 \neq 0$, $\operatorname{Im} z_2 \neq 0$,

$$(\hat{H}_0 - z_1)^{-1} - (\hat{H}_0 - z_2)^{-1} = (z_1 - z_2)(\hat{H}_0 - z_1)^{-1}(\hat{H}_0 - z_2)^{-1}$$

we get, for $\operatorname{Im} z \neq 0$,

$$(z \pm i)\int_T d\omega\, G(\omega, \omega_1, z)\cdot G(\omega, \omega_2, \mp i) = G(\omega_1, \omega_2, z) - G(\omega_1, \omega_2, \mp i)$$

which together with the decomposition of $G(\omega, \omega', z)$, obtained from the spectral representation of $\hat{H}_0$,

$$G(\omega, \omega', z) = \sum_{n,m} \frac{e^{in(\omega_1 - \omega_1')} e^{im(\omega_2 - \omega_2')}}{E_{n,m} - z} \tag{2.12}$$

[recalling that $E_{n,m}$ are the eigenvalues of $\hat{H}_0$; see (2.2), (2.5)], leads to

$$\Lambda(z, \Theta) = (1 - e^{i\Theta}) \sum_{n,m} \left(\frac{-1}{E_{n,m} - z} + \frac{E_{n,m}}{E_{n,m}^2 + 1} + \frac{\sin \Theta}{1 - \cos \Theta} \frac{1}{E_{n,m}^2 + 1} \right) \tag{2.13}$$

Hence for $\Theta \neq 0$ the zeros of the function $\Lambda(z, \Theta)$ are equal to the zeros of the meromorphic function $\xi(z, \Theta) \equiv (\lambda(z, \Theta))^{-1}$,

$$\xi(z, \Theta) = \sum_{n,m} \left(\frac{-1}{E_{n,m} - z} + \frac{E_{n,m}}{E_{n,m}^2 + 1} \right) + \frac{\sin \Theta}{1 - \cos \Theta} \sum_{n,m} \frac{1}{E_{n,m}^2 + 1} \tag{2.14}$$

For $\Theta = \pi$ (infinite coupling) we get

$$\xi(z) \equiv -\xi(z, \pi) = \sum_{n,m} \left(\frac{1}{E_{n,m} - z} - \frac{E_{n,m}}{E_{n,m}^2 + 1} \right) \tag{2.15}$$

or in the one-index notation (with ordered eigenvalues E_i)

$$\xi(z) = \sum_n \left(\frac{1}{E_n - z} - \frac{E_n}{E_n^2 + 1} \right) \tag{2.16}$$

We remark that $z \to \xi(z, \Theta)$ is monotone increasing in all intervals (E_i, E_{i+1}), $E_i \geqslant 0$. Hence we have the following result.

Proposition 4. For $\Theta \neq 0$ the eigenvalues of $\hat{H}_\Theta$ coincide with the zeros of the meromorphic function $\xi(z, \Theta)$ given by (2.14). For $\Theta = 0$, $\hat{H}_\Theta$ coincides with the free Hamiltonian $\hat{H}_0$. For $\Theta = \pi$ the function $\xi(z, \Theta)$ simplifies according to (2.15) or (2.16). $E_{n,m}$ are the poles of $\xi(z, \Theta)$, for $\Theta \neq 0$.

For simplicity in the following we shall only discuss the case $\Theta = \pi$. We shall denote the eigenvalues of the corresponding Hamiltonian $\hat{H}_\pi$ by z_n. We shall assume, according to the hypothesis discussed in Section 1, that $E_{n,m}$ are Poisson distributed. From this information we shall draw conclusions on the zeros z_n of $\xi(z)$.

3. DISTRIBUTION OF ZEROS AND EIGENVALUES OF THE HAMILTONIAN

Let us now investigate the distribution of zeros z_n of the function $\xi(z)$ given by (2.16) in more detail, assuming that the poles E_n are Poisson distributed. We first replace the function $\xi(z)$ by its finite sum approximation $\xi_N(z)$,

$$\xi_N(z)=\sum_{n=1}^{N}\left(\frac{1}{E_n-z}-\frac{E_n}{E_n^2+1}\right) \tag{3.1}$$

$\xi_N(z)$ converges pointwise to $\xi(z)$ as $N\to\infty$.

Let $z_1,..., z_n$ denote the ordered roots of $\xi_N(z)$: $\xi_N(z_i)=0$, $z_i\leqslant z_{i+1}$.

Our aim is to find their joint distribution provided the numbers $E_1\cdots E_N$ are given by a Poisson process.

We express $\xi_N(z)$ as

$$\xi_N(z)=\tilde{\xi}_N(z)-C_N \tag{3.2}$$

with

$$\begin{aligned}\tilde{\xi}_N(z)&=\sum_{n=1}^{N}\frac{1}{E_n-z}\\ C_N&=\sum_{n=1}^{N}\frac{E_n}{E_n^2+1}\end{aligned} \tag{3.3}$$

The function can be now expressed as

$$\xi_N(z)=\frac{P_1(z)}{P_2(z)} \tag{3.4}$$

where $P_1(z)$, $P_2(z)$ are polynomials of the Nth order,

$$\begin{aligned}P_2(z)&=\prod_{i=1}^{N}(E_i-z)\\ P_1(z)&=\left(\sum_{i=1}^{N}\prod_{\substack{j=1\\ j\neq i}}^{N}(E_j-z)\right)-C_NP_2(z)\end{aligned} \tag{3.5}$$

The distribution of roots of the function $\xi_N(z)$ coincides with the distribution of roots of the polynomial $P_1(z)$. Let us write $P_1(z)$ as

$$P_1(z)=a_1z^N+a_2z^{N-1}+\cdots+a_Nz+a_{N+1} \tag{3.6}$$

with

$$\begin{aligned}
a_1 &= -C_N(-1)^N \\
a_2 &= (-1)^{N-1}\,[-C_N(E_1+E_2+\cdots+E_N)+N] \\
a_3 &= (-1)^{N-2}\left[-C_N\left(\sum_{i<j}^{N} E_iE_j\right)+(N-1)(E_1+E_2+\cdots+E_N)\right] \\
&\vdots \\
a_k &= (-1)^{N+1-k}\left[-C_N\left(\sum_{i_1<i_2<\cdots<i_k} E_{i_1}E_{i_2}\cdots E_{i_k}\right)\right. \\
&\qquad \left.+(N-k+1)\left(\sum_{i_1<i_2<\cdots<i_{k-1}} E_{i_1}E_{i_2}\cdots E_{i_{k-1}}\right)\right]
\end{aligned} \tag{3.7}$$

Decomposing on the other hand $P_1(z)$ as

$$P_1(z) = a_1(z-z_1)(z-z_2)\cdots(z-z_N) \tag{3.8}$$

and comparing the corresponding coefficients, we get equations connecting the roots $\mathbf{z} \equiv (z_1 \cdots z_N)$ and the poles $\mathbf{E} \equiv (E_1 \cdots E_N)$,

$$F_i(\mathbf{z}, \mathbf{E}) = 0, \qquad i = 1,\ldots, N \tag{3.9}$$

with

$$\begin{aligned}
F_1(\mathbf{z};\mathbf{E}) &= E_1+E_2+\cdots+E_N-(z_1+z_2+\cdots+z_N)-\frac{N}{C_N} \\
F_2(\mathbf{z};\mathbf{E}) &= \sum_{i<j} E_iE_j-\left(\frac{N-1}{C_N}\right)(E_1+\cdots+E_N)-\left(\sum_{i<j} z_iz_j\right) \\
&\vdots \\
F_k(\mathbf{z};\mathbf{E}) &= \sum_{i_1<\cdots<i_k} E_{i_1}\cdots E_{i_k}+\cdots-\left(\frac{N+1-k}{C_N}\right) \\
&\qquad \times\left(\sum_{i_1<\cdots<i_{k-1}} E_{i_1}\cdots E_{i_{k-1}}\right)-\left(\sum_{i_1<\cdots<i_k} z_{i_1}\cdots z_{i_k}\right)
\end{aligned} \tag{3.10}$$

Solving this system of equations, we obtain the roots z_i as functions of E_i,

$$z_i = z_i(E_1 \cdots E_N) \tag{3.11}$$

Assume now the $E_1,\ldots, E_N$ are given by a Poisson process starting from the origin and having parameter normalized to 1. Then we have for the joint probability density that the ith root has the value z_i,

$$P(z_1 \cdots z_N) = \frac{D(E_1 \cdots E_N)}{D(z_1 \cdots z_N)}\, e^{-E_N} \tag{3.12}$$

where $D(E_1 \cdots E_N)/D(z_1 \cdots z_N)$ is the Jacobian of the transformation (3.11).

To compute the Jacobian $D(E_1 \cdots E_N)/D(z_1 \cdots z_N)$, we use the identity

$$\frac{D(E_1 \cdots E_N)}{D(z_1 \cdots z_N)} = \frac{D(F_1 \cdots F_N)/D(z_1 \cdots z_N)}{D(F_1 \cdots F_N)/D(E_1 \cdots E_N)} \tag{3.13}$$

which leads to

$$\frac{D(E_1 \cdots E_N)}{D(z_1 \cdots z_N)} = \frac{\prod_{1 \leqslant i < j \leqslant N} |(z_i - z_j)|}{\prod_{1 \leqslant i < j \leqslant N} |(E_i - E_j)|} \tag{3.14}$$

We note that $D(F_1 \cdots F_N)/D(z_1 \cdots z_N)$ is nothing but the well-known Vandermonde determinant; $D(F_1 \cdots F_N)/D(E_1 \cdots E_N)$ can also be evaluated when one realizes that the corresponding matrix contains rows which are linear combinations of rows of a matrix which has again the Vandermonde form.

Summarizing these results, we find that the joint distribution of roots of the function $\xi_N(z)$ is given by

$$P(z_1 \cdots z_N) = \frac{\prod_{1 \leqslant i < j \leqslant N} |(z_i - z_j)|}{\prod_{1 \leqslant i < j \leqslant N} |(E_i - E_j)|} e^{-E_N} \tag{3.15}$$

Hence we have proven the following result.

Proposition 5. Let

$$\xi_N(z) = \sum_{n=1}^{N} \left(\frac{1}{E_n - z} - \frac{E_n}{E_n^2 + 1} \right), \qquad z \in \mathbb{C}$$

where E_n are the eigenvalues of $\hat{H}_0$, with $\hat{H}_0$ being the Hamiltonian of the quantum mechanical system defined in Section 2, whose classical counterpart is integrable. Assume E_n, $n = 1, \ldots, N$, are Poisson distributed with parameter 1. Then the joint distribution $P(z_1, \ldots, z_N)$ of the roots of ξ_N is given by

$$P(z_1 \cdots z_N) = \frac{\prod_{1 \leqslant i < j \leqslant N} |(z_i - z_j)|}{\prod_{1 \leqslant i < j \leqslant N} |(E_i - E_j)|} e^{-E_N}$$

Remark. From Proposition 4, the zeros of $\xi(z)$ coincide with the eigenvalues of the perturbed Hamiltonian $\hat{H}_\pi$. Since $\xi_N(z) \to \xi(z)$, the roots of ξ_N given in Proposition 5 yield information on the zeros of $\xi(z)$, hence on the eigenvalues of the perturbed Hamiltonian $\hat{H}_\pi$. Comparing this result with the predictions of the RMT, we see that the distribution we found does not coincide with the one expected from RMT, which should be a Wishart distribution.[11]

4. ASYMPTOTICS OF THE LEVEL SPACING DISTRIBUTION BETWEEN ZEROS

Let $z_1 \leqslant z_2 \leqslant \cdots \leqslant z_n \leqslant z \cdots$ be the zeros of the function $\xi(z)$, i.e.,

$$\xi(z_i) = 0, \qquad i \in \mathbb{N} \tag{4.1}$$

defined by [see (2.15)]

$$\xi(z) = \sum_{n=1}^{\infty} \left(\frac{1}{E_n - z} - \frac{E_n}{E_n^2 + 1} \right) \tag{4.2}$$

Introducing $s_i = z_{i+1} - z_i$, the spacings between zeros which are neighbors, we define the level spacing probability $P(s)$, $s \geqslant 0$, in such a way that

$$\begin{aligned} &P(s)\, ds \text{ is the probability that } s_i \text{ belong} \\ &\quad \text{to the interval } [s; s+ds], \quad \text{for some} \quad i \in \mathbb{N} \end{aligned} \tag{4.3}$$

$P(s)\, ds$ then also gives information about the distributions of eigenvalues for $\hat{H}_\pi$, according to Proposition 4. We would like to investigate $P(s)$ for small s. The aim is to show that

$$c_1 s \leqslant P(s) \leqslant c_2 s \tag{4.4}$$

for all $s \leqslant s_0$, for some $s_0 > 0$, where c_1, c_2 are constants.

It is clear that $\int_B P(s)\, ds \geqslant \int_B \tilde{P}(s)\, ds$, where $\int_B \tilde{P}(s)\, ds$ is the probability to find three poles within the closed interval B (since if we have three poles in B, then we have surely at least two zeros in B—the zeros lying between the poles). This implies $P(s) \geqslant \tilde{P}(s)$. The poles E_n are, however, by assumption, Poisson distributed, and hence

$$\tilde{P}(s) = s \cdot e^{-s} \tag{4.5}$$

We have therefore the asymptotic estimate for $s \downarrow 0$:

$$P(s) \geqslant s \cdot e^{-s} = \tilde{P}(s) \tag{4.6}$$

The next step is to give an estimate the other way around, namely that

$$P(s) \leqslant k \cdot s \qquad \text{as} \quad s \downarrow 0 \tag{4.7}$$

with k being some positive constant. In order to do this, we will investigate the probability hat two zeros are contained in an interval of length s. It is clear that if two zeros are contained in this interval, then at least *one* pole is also contained there (since between two zeros there is exactly one pole).

We will denote this pole by E_{n_0} and place it in the center of the interval. Our aim is to estimate the quantities

$$\begin{aligned} \Delta_+ &\equiv E_{n_0} - z_{n_0} \\ \Delta_- &\equiv z_{n_0+1} - E_{n_0} \end{aligned} \tag{4.8}$$

In order to do this, we write $\xi(z)$ as

$$\xi(z) = \frac{1}{E_{n_0} - z} + \tilde{\xi}(z) \tag{4.9}$$

with

$$\tilde{\xi}(z) = -\frac{E_{n_0}}{E_{n_0}^2 + 1} + \sum_{n \neq n_0} \left(\frac{1}{E_n - z} - \frac{E_n}{E_n^2 + 1} \right) \tag{4.10}$$

It is reasonable to decompose $\tilde{\xi}(z)$ into the positive and negative parts

$$\tilde{\xi}(z) = \xi_+(z) - \xi_-(z) \tag{4.11}$$

with $\xi_\pm(z)$ defined by

$$\begin{aligned} \xi_+(z) &= \sum_{E_n \geqslant z} \left(\frac{1}{E_n - z} - \frac{E_n}{E_n^2 + 1} \right) \\ \xi_-(z) &= -\frac{E_{n_0}}{E_{n_0}^2 + 1} - \sum_{E_n < z} \left(\frac{1}{E_n - z} - \frac{E_n}{E_n^2 + 1} \right) \end{aligned} \tag{4.12}$$

Using this definition, we find

$$\begin{aligned} \frac{1}{\Delta_+} &= -\tilde{\xi}(z_{n_0}) = \xi_-(z_{n_0}) - \xi_+(z_{n_0}) \leqslant \xi_-(z_{n_0}) \leqslant \xi_-\left(E_{n_0} - \frac{s}{2}\right) \\ \frac{1}{\Delta_-} &= \tilde{\xi}(z_{n_0+1}) = \xi_+(z_{n_0+1}) - \xi_-(z_{n_0+1}) \leqslant \xi_+(z_{n_0+1}) \leqslant \xi_+\left(E_n + \frac{s}{2}\right) \end{aligned} \tag{4.13}$$

(Here we used the fact that ξ_+ and ξ_- are monotonically increasing/decreasing functions, respectively, which can be easily verified by taking derivatives, exploiting uniform convergence of the series defining $\xi_\pm$.)

Hence

$$\begin{aligned} \Delta_+ &\geqslant \frac{1}{\xi_-(E_{n_0} - s/2)} \\ \Delta_- &\geqslant \frac{1}{\xi_+(E_{n_0} + s/2)} \end{aligned} \tag{4.14}$$

and we can write for the level spacing

$$z_{n0+1}-z_{n0}=\Delta_{+}+\Delta_{-}\geqslant\frac{1}{\xi_{-}(E_{n_0}-s/2)}+\frac{1}{\xi_{+}(E_{n_0}+s/2)} \tag{4.15}$$

For the probabilities we find

$$\begin{aligned}P\{z_{n0+1}-z_{n0}\leqslant s\}&=P\{\Delta_{+}+\Delta_{-}\leqslant s\}\\&\leqslant P\left\{\frac{1}{\xi_{+}(E_{n_0}+s/2)}+\frac{1}{\xi_{-}(E_{n_0}-s/2)}\leqslant s\right\}\end{aligned} \tag{4.16}$$

From the assumption on the E_n that they be Poisson distributed, we have that the variables ξ_+ and ξ_- are independent (the corresponding sums run over different indices). There exists a value s_0, depending on n_0, such that

$$\xi_{+}\left(E_{n_0}+\frac{s}{2}\right)\geqslant\frac{1}{s}\qquad\text{for}\quad 0\leqslant s\leqslant s_0 \tag{4.17}$$

only if

$$\frac{1}{E_{n0+1}-E_{n0}-s/2}\geqslant\frac{1}{s} \tag{4.18}$$

(all the other terms in the sum lead to smaller contributions), which means

$$E_{n0+1}-E_{n0}\leqslant\tfrac{3}{2}s \tag{4.19}$$

Therefore, using the fact that the $\{E_n\}$ are Poisson distributed, we have, for $s\downarrow 0$

$$P\left\{\frac{1}{\xi_{+}}\leqslant s\right\}=P\left\{\xi_{+}\geqslant\frac{1}{s}\right\}=P\left\{E_{n0+1}-E_{n0}\leqslant\frac{3}{2}s\right\}+O(s^2) \tag{4.20}$$

Similarly,

$$P\left\{\frac{1}{\xi_{-}}\leqslant s\right\}=P\left\{\xi_{-}\geqslant\frac{1}{s}\right\}=P\left\{E_{n0}-E_{n0-1}\leqslant\frac{3}{2}s\right\}+O(s^2) \tag{4.21}$$

The poles E_n are, however, given according to our assumption by a Poisson process and therefore

$$\begin{aligned}P\left\{\xi_{+}\geqslant\frac{1}{s}\right\}&=\frac{3}{2}s+O(s^2)\\P\left\{\xi_{-}\geqslant\frac{1}{s}\right\}&=\frac{3}{2}s+O(s^2)\end{aligned} \tag{4.22}$$

For the level-spacing probability we get

$$P\{z_{n_0+1} - z_{m_0} \leqslant s\} \leqslant P\left\{\frac{1}{\xi_+} + \frac{1}{\xi_-} \leqslant s\right\} = \frac{9}{8}s^2 + O(s^3) \tag{4.23}$$

which leads to

$$P(s) \leqslant \tfrac{9}{4}s \qquad \text{as} \quad s \to 0 \tag{4.24}$$

Summarizing the above estimates, we have for the level-spacing distribution

$$s \leqslant P(s) \leqslant \tfrac{9}{4}s \qquad \text{as} \quad s \to 0 \tag{4.25}$$

Hence we have proven the following result.

Theorem 6. Assume the eigenvalues E_n of the Hamiltonian $\hat{H}_0$ corresponding to the classical integrable system described in Section 2 are Poisson distributed. Let $P(s)\,ds$ be the probability that some of the spacings $z_{i+1} - z_i$ between two successive zeros of the function $\xi(z)$ giving the eigenvalues of the interacting Hamiltonian $\hat{H}_\pi$ belong to the interval $[s; s+ds]$. Then one has $P(s) \geqslant se^{-s}$, and $s \leqslant P(s) \leqslant (9/4)s$, as $s \downarrow 0$.

Remark. The estimate for $s \downarrow 0$ is in agreement with RMT. The situation for $s \to \infty$ is different. It is known that the RMT gives $c_1 e^{-s^2} \leqslant P(s) \leqslant c_2 e^{-s^2}$ for some constants c_1, c_2 as $s \to \infty$ (see, e.g., ref. 11). This conflicts for large s with the above estimate $P(s) \geqslant e^{-s}$, which holds for large s. Hence, for large s the level-spacing probability certainly does not coincide with the one predicted by the RMT.

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